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A Characterization of Locally Artinian Modules

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This paper continues the study of an R-module M through properties of the category $\sigma[M]$ of submodules of M-generated modules. It is shown that a self-projective M is locally artinian if and only if every cyclic module in $\sigma[M]$ is a direct sum of an M-injective and a finitely cogenerated module. From this we derive that a ring T with local units is locally left artinian if and only if every module in T-*Mod* is a direct sum of a T-injective module and a finitely cogenerated module. For rings with unity this was proved in Huynh–Dung [3] applying Osofsky's observation about left-injective regular rings R: For an infinite set of orthogonal idempotents $\{e_\lambda\}_\Lambda$ the factor module $R/\sum_\Lambda Re_\lambda$ is not injective. Combined with a categorical equivalence Huynh–Dung's result is also used in our own proof.

As a consequence we observe that a module M is semisimple if and only if it is self-projective and every cyclic module in $\sigma[M]$ is M-injective. This enables us to extend a famous result of Osofsky's for rings with unit to the more general case: A ring T with local units is semisimple if and only if every cyclic T-module is injective.

Some more applications concerning SI-rings and decompositions of modules are given.

Let R be an associative ring with unity and R-*Mod* the category of unital left R-modules. For $M \in R$-*Mod* we denote by $\sigma[M]$ the full subcategory of R-*Mod* whose objects are submodules of M-generated modules. M is called *self-projective* (*self-injective*) if it is M-projective (M-injective). For basic definitions see [1, 9].

As examples for the present considerations one may always have in mind the following situations (e.g., [9, 15.10, 18.11]):

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1. *The subcategories* $\sigma[\mathbb{Q}/\mathbb{Z}]$ *and* $\sigma[\mathbb{Z}_{p^\infty}]$ *(p prime number) of* $\mathbb{Z}$*-Mod are the torsion* $\mathbb{Z}$*-modules (abelian groups) and the p-groups.*

Both categories do not have any projective object.

$\bigoplus \{\mathbb{Z}_{p^k} \mid k \in \mathbb{N}, p$ *all prime numbers*$\}$ *is a (sub-) generator in* $\sigma[\mathbb{Q}/\mathbb{Z}]$ *and* $\bigoplus \{\mathbb{Z}_{p^k} \mid k \in \mathbb{N}\}$ *is a generator in* $\sigma[\mathbb{Z}_{p^\infty}]$.

The $\mathbb{Z}_{p^k}$ *are self-projective* $\mathbb{Z}$*-modules.*

We begin with a technical observation:

2. LEMMA. *Let* M *be a cyclic, self-projective, and self-injective left* R*-module with* $\mathrm{Rad}(M) = 0$. *Assume that every cyclic submodule of* M *is a direct sum of an* M*-injective and a finitely cogenerated module. Then*

(i) *Every cyclic submodule of* M *is a direct summand.*

(ii) *The functor* $\mathrm{Hom}_R(M, -): \sigma[M] \to \mathrm{End}(M)$*-Mod is an equivalence.*

(iii) $\mathrm{End}(M)$ *is left self-injective and (von Neumann) regular.*

Proof. (i) Since $\mathrm{Rad}(M) = 0$ there are no superfluous submodules in M and hence every simple submodule of M is a direct summand and M-injective. As a consequence, every finitely generated semisimple submodule of M is a direct summand.

Let $N = E \oplus C$ be a cyclic submodule of M with E M-injective and C finitely cogenerated. Then C is semisimple and a direct summand of M. This implies that $E \oplus C$ is M-injective and a direct summand of M.

(ii) By (i), M is a self-projective self-generator and therefore a projective generator in $\sigma[M]$ which means that $\mathrm{Hom}(M, -)$ defines an equivalence.

(iii) Follows immediately from (i) and (ii).

An R-module M is called *locally artinian*, if every finitely generated submodule of M is artinian. This is equivalent to the fact that every finitely generated (cyclic) module in $\sigma[M]$ is artinian (e.g., [9, Sect. 31]). Generalizing the characterization of left artinian rings in Huynh–Dung [3] we prove:

3. THEOREM. *For an* R*-module* M *the following are equivalent:*

(a) M *is locally artinian;*

(b) *every cyclic module in* $\sigma[M]$ *is a direct sum of a self-projective,* M*-injective module and a finitely cogenerated module.*

If M *is a direct sum of cyclic self-projective modules then* (a) *and* (b) *are equivalent to:*

(c) *every cyclic module in* $\sigma[M]$ *is a direct sum of an* M*-injective module and a finitely cogenerated module.*

Proof. (a) $\Rightarrow$ (b), (c) is clear since for locally artinian M every cyclic module in $\sigma[M]$ is finitely cogenerated.

(b) $\Rightarrow$ (a) We show that every cyclic module X in $\sigma[M]$ is finitely cogenerated. This obviously implies that the cyclic modules are artinian and M is locally artinian.

By hypothesis we have

$$X = Y \oplus F \tag{$*$}$$

with ${}_RY$ self-projective and M-injective and ${}_RF$ finitely cogenerated.

We first consider the case Rad $X = 0$. Then Rad $F = 0$ and F is a finitely generated semisimple module. Also, Rad $Y = 0$ and, by Lemma 2, we have an equivalence

$$\mathrm{Hom}(Y, -) \colon \sigma[Y] \to S-Mod \qquad \text{for} \quad S = \mathrm{End}(Y).$$

Since Y is cyclic every cyclic left module L in S-*Mod* is of the form $L \simeq \mathrm{Hom}(Y, Y')$ for a (cyclic) factor module Y' of Y. Hence L is a direct sum of a (self-projective) S-injective module and a finitely cogenerated module. By Theorem 1.1 in Huynh–Dung [3], ${}_SS$ is artinian and because of the given equivalence ${}_RY$ is artinian (and semisimple). This implies that $X = Y \oplus F$ is finitely cogenerated (and semisimple).

Now consider the general case Rad $X \neq 0$. In $(*)$, ${}_RF$ is finitely cogenerated. Since Rad Y does not contain non-zero Y-injective submodules, every cyclic submodule of Rad Y is finitely cogenerated and hence Soc(Rad Y) is essential in Rad Y. By the above argument $Y/\mathrm{Rad}\, Y$ is semisimple and we deduce that Soc Y is essential in Y.

Soc Y is finitely generated: Let k be the (composition) length of $Y/\mathrm{Rad}\, Y$ and suppose that the length of Soc Y is greater than k. Then

$$Y = E(S_1) \oplus \cdots \oplus E(S_k) \oplus Y',$$

where $E(S_i)$ denotes the Y-injective hull of a simple submodule $S_i \subset Y$ and Y' is a non-zero summand of Y. From this we get

$$\mathrm{Rad}\, Y = \mathrm{Rad}\, E(S_1) \oplus \cdots \oplus \mathrm{Rad}\, E(S_k) \oplus \mathrm{Rad}\, Y',$$

where Rad $E(S_i) \neq E(S_i)$ for every $i \leqslant k$ and Rad $Y' \neq Y'$. We conclude that the length of $Y/\mathrm{Rad}\, Y$ is at least $k+1$, a contradiction.

Hence Soc Y is finitely generated and Y and X are finitely cogenerated.

(c) $\Rightarrow$ (a) First assume that M is cyclic and self-projective. Then $M/\mathrm{Rad}\, M$ is also cyclic and self-projective. By the first part of the proof (a)$\Rightarrow$(b) we obtain that $M/\mathrm{Rad}\, M$ is semisimple. The second part of this proof shows that every factor module of M is finitely cogenerated; i.e., M is artinian.

Now consider $M = \bigoplus_\Lambda M_\lambda$ with cyclic self-projective modules M_λ. Condition (c) implies that every cyclic module in $\sigma[M_\lambda]$ is a direct sum of an M_λ-injective module and a finitely cogenerated module. The above arguement tells us that the M_λ's are artinian and therefore M is locally artinian.

Remark. A module $N \in \sigma[M]$ is called a *subgenerator in* $\sigma[M]$ if $\sigma[M] = \sigma[N]$. The condition on M in the second part of the theorem could be replaced by the same condition for a suitable subgenerator in $\sigma[M]$. For example, though $\mathbb{Q}/\mathbb{Z}$ is not a direct sum of cyclic self-projective modules we have $\sigma[\mathbb{Q}/\mathbb{Z}] = \sigma[\bigoplus \{\mathbb{Z}_{p^k} \mid k \in \mathbb{N},\ p \text{ prime}\}]$ (compare 1).

Generally, for every associative ring T without unit the category $\sigma[{}_TT]$ is a good module category to investigate properties of T. We consider a special case: T is called a ring *with local units* if every finite subset of T is contained in a subring of the form eTe for an idempotent $e \in T$ (see [2]). For such rings the category $\sigma[{}_TT]$ consists of all T-modules L with $TL = L$ and is denoted by *T-Mod.*

The set $\{Te \mid e^2 = e \in T\}$ is a generating set of cyclic T-projective modules and as a special case of the above theorem we can formulate:

4. COROLLARY. *A ring T with local units is locally artinian if and only if every cyclic module in T-Mod is a direct sum of a T-injective module and a finitely cogenerated module.*

Modules over rings T without units are also studied in Tominaga [7]. He calls a left T-module M *s-unital* if $m \in Tm$ for every $m \in M$. If ${}_TT$ is s-unital the category $\sigma[{}_TT]$ consists of all s-unital T-modules and T is a generator in $\sigma[{}_TT]$. Obviously, rings with local units are left and right s-unital.

An R-module M is called *co-semisimple* if every simple module in $\sigma[M]$ is M-injective. Various characterizations of co-semisimple modules are given in [8]. They imply the description of *left s-unital V-rings* in [7, Theorem 4].

It was proved in Osofsky [5] that a ring whose cyclic (left) modules are injective has to be left semisimple. What can be said about modules M for which all cyclic modules in $\sigma[M]$ are M-injective?

5. COROLLARY. *For an R-module M the following are equivalent*:

(a) *M is semisimple*;

(b) *every cyclic module in* $\sigma[M]$ *is self-projective and* M*-injective*;

(c) M *is a direct sum of self-projective modules and every cyclic module in* $\sigma[M]$ *is* M*-injective.*

Proof. (a) $\Rightarrow$ (b), (c) is trivial since for semisimple M every module is projective and injective in $\sigma[M]$.

(b) $\Rightarrow$ (a) Assume the cyclic modules in $\sigma[M]$ to be self-projective and M-injective. Then M is co-semisimple and every module in $\sigma[M]$ has zero radical. By our Theorem, M is locally artinian and hence every (finitely generated) module in $\sigma[M]$ is semisimple.

(c) $\Rightarrow$ (a) We may assume that M is self-projective. Suppose the cyclic modules in $\sigma[M]$ are M-injective. Then M is co-semisimple. The cyclic submodules $L \subset M^k$, $k \in \mathbb{N}$, form a generating set for $\sigma[M]$ and are direct summands and M-projective. We know from our Theorem that M is locally artinian and hence M is semisimple.

6. COROLLARY. *A ring* T *with local units is left semisimple if and only if every cyclic module in* T*-Mod is* T*-injective.*

A left s-unital ring T *is left semisimple if and only if it is self-projective and every (s-unital) left* T*-module is* T*-injective.*

Following Goodearl [4] a ring R is called *left SI-ring* if every singular left R-module is injective (equivalently: semisimple). In Smith [6], R is called *left RIC-ring* if every *cyclic* singular left R-module is injective. It is shown in [6, Theorem 3.9] that commutative *RIC*-rings are *SI*-rings. Obviously, any cyclic module over a commutative ring is self-projective. Hence the following application of Corollary 5 extends the result just mentioned:

7. COROLLARY. *For a ring* R *the following conditions are equivalent*:

(a) R *is a left SI-ring*;

(b) R *is a left RIC-ring and every cyclic singular left* R*-module is self-projective.*

Proof. We only have to show that (b) implies (a). Let M be a singular left R-module. Then every cyclic module in $\sigma[M]$ is singular and hence M-injective and self-projective by (b). According to Corollary 5, M has to be semisimple.

An R-module L is called *finite*, if L is a finite set. Here we call L *locally finite*, if every finitely generated submodule of L is finite. For example, every torsion $\mathbb{Z}$-module is locally finite.

8. COROLLARY. *Let M be an R-module. Assume that every cyclic module in $\sigma[M]$ is a direct sum of a self-projective, M-injective module and a finite module. Then M is a direct sum of a semisimple and a locally finite module.*

Proof. By the Theorem, M is locally artinian. Let S denote the sum of all infinite minimal submodules of M and F the sum of all finite submodules of M. Then S is semisimple, F is locally finite, and $S \cap F = 0$.

We show that $M = S + F$: Suppose there is an $x \in M$ and $x \notin S + F$. Then Rx certainly is not finite. By hypothesis, $Rx = U_1 \oplus \cdots \oplus U_k \oplus L$ with infinite indecomposable M-injective U_i's and finite $L \subset F$. Since the U_i's have to be simple, we have $U_1 \oplus \cdots \oplus U_k \subset S$ and $Rx \subset S + F$, a contradiction.

9. COROLLARY. *Let M be an R-module. Assume that every cyclic module in $\sigma[M]$ is a direct sum of a self-projective M-injective module and a semisimple module. Then $M = \bigoplus_\Lambda M_\lambda$ where each M_λ is either simple or M_λ is M-injective, indecomposable, and of length* 2.

Proof. By our Theorem, M is locally artinian. Let X be a non-zero cyclic submodule of M. We may write

$$X = E(T_1) \oplus \cdots \oplus E(T_k) \oplus S_1 \oplus \cdots \oplus S_t,$$

where T_i and S_j are simple modules and the $E(T_i)$ are the M-injective hulls of $T_i \neq E(T_i)$. Also, the $E(T_i)$'s are self-projective. Take $x_i \in E(T_i)$, $x_i \notin T_i$. Then $T_i \subset Rx_i$, Rx_i is uniform and—by hypothesis—M-injective. This implies $Rx_i = E(T_i)$ and the $E(T_i)$'s have length 2. As a consequence, M is locally noetherian.

Choose $\{X_\alpha\}_A$ as a maximal independent family of indecomposable, M-injective submodules $X_\alpha \subset M$. Then $N = \bigoplus_A X_\alpha$ is M-injective and $M = N \oplus M'$. Now it can easily be checked that M' has to be semisimple.

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